Characters of A_{n-1} Hecke algebras at roots of unity

1992 J. Phys. A: Math. Gen. 25 L789

(http://iopscience.iop.org/0305-4470/25/13/006)

J. Phys. A: Math. Gen. 25 (1992) L789–L797. Printed in the UK

LETTER TO THE EDITOR

Characters of A_{n-1} Hecke algebras at roots of unity

C J Cummins† and R C King‡

† Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada

‡ Faculty of Mathematical Studies, University of Southampton, Southampton SO9 5NH, UK

Received 23 March 1992

Abstract. A Frobenius formula is presented for the characters, $\phi_\rho^\sigma(q)$, of the irreducible representations, $\pi_{[\sigma]}$, of the A_{n-1} Hecke algebras $H_n(q)$ labelled by (m,k)-standard partitions, σ, where q is a primitive pth root of unity with $p = m + k$. Using this result the characters $\phi_\rho^\sigma(q)$ may be expressed as linear combinations of the characters, $\chi_\rho^\lambda(q)$, of representations which are irreducible when q is not a root of unity. The appropriate linear combinations are found by using fusion modification rules. For $1 \leqslant n \leqslant 5$ all the remaining characters of irreducible representations of $H_n(q)$ are then found, allowing a complete tabulation to be made for $2 \leqslant p \leqslant 5$, along with the corresponding decomposition matrices.

The complex Hecke algebra $H_n(q)$, with q an arbitrary but fixed complex parameter, is generated by g_i with $i = 1, 2, \ldots, n-1$ subject to the relations

$$g_i^2 = (q-1)g_i + q \qquad \text{for} \quad i = 1, 2, \ldots, n-1 \tag{1a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for} \quad i = 1, 2, \ldots, n-2 \tag{1b}$$

$$g_i g_j = g_j g_i \qquad \text{for} \quad |i-j| \geqslant 2. \tag{1c}$$

For $q = 1$ these relations are exactly those appropriate to the group algebra of the symmetric group S_n with g_i replaced by s_i for $i = 1, 2, \ldots, n-1$, where s_i is the transposition $(i, i+1)$. Every permutation s in S_n can be expressed as a word of minimal length, $l(s)$, in the generators s_i. Although such an expression is not unique a canonical form may be specified (see for example King and Wybourne 1992). Given such a form there exists a map h from S_n to $H_n(q)$ such that $h(s_i) = g_i$ and, more generally, $h(s) = g_{i_1} g_{i_2} \cdots g_{i_{l(s)}}$ for any $s = s_{i_1} s_{i_2} \cdots s_{i_{l(s)}} \in S_n$. The set of all $h(s)$ for $s \in S_n$ is a basis of $H_n(q)$ (Gyoja 1986, Jones 1987).

As explained elsewhere (Ram 1991, King and Wybourne 1992), any trace of an arbitrary element, x, of $H_n(q)$ can be expressed as a linear sum of traces of certain words, $v = h(s)$, which are said to be minimal, in that they contain no generator g_i more than once. Moreover, each minimal word $v = h(s)$ can be assigned to the connectivity class, (ρ), labelled by the partition ρ which specifies the conjugacy class of $s \in S_n$.

If q is not a root of unity, the inequivalent finite-dimensional complex irreducible representations, $\pi_{\{\lambda\}}$, of $H_n(q)$ are labelled by the elements λ of P_n, the set of partitions of n. Each such partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ specifies a Young diagram F^λ consisting of n boxes arranged in left-adjusted rows of length $\lambda_1, \lambda_2, \ldots$. The corresponding column lengths $\lambda'_1, \lambda'_2, \ldots$ serve to define the partition λ' conjugate to λ. The representation $\pi_{\{\lambda\}}$ may be realized (Dipper and James 1986) in the form of a module $V^{\{\lambda\}}$ with basis labelled by the set T_λ of all standard Young tableaux t_i^λ of shape λ, where $i = 1, 2, \ldots, f^\lambda$. The dimension, f^λ, of $\pi_{\{\lambda\}}$ coincides with that of the irreducible representation of the symmetric group, S_n, also labelled by λ.

A number of authors (Carter 1986, Vershik and Kerov 1989, Ram 1991, King and Wybourne 1990, 1992, Van der Jeugt 1991) have independently derived rather simple procedures or formulae for the evaluation of the characters $\chi_\rho^\lambda(q)$ of these irreducible representations $\pi_{\{\lambda\}}$. In particular (King and Wybourne 1990, 1992) they may be obtained from the Frobenius formula (cf Macdonald 1979, p60)

$$p_\rho(x;q) = \sum_{\lambda \in P_n} \chi_\rho^\lambda(q) s_\lambda(x) \tag{2}$$

where P_n denotes the set of all partitions of n, $s_\lambda(x)$ is the S-function labelled by λ and

$$p_\rho(x;q) = p_{\rho_1}(x;q)\ p_{\rho_2}(x;q)\ \cdots \tag{3}$$

with

$$p_r(x;q) = \sum_{\substack{a,b=0 \\ a+b+1=r}}^{r-1} (-1)^b q^a s_{(a+1,1^b)}(x). \tag{4}$$

As pointed out by Ram (1991) $p_r(x;q)$ is nothing other than a Hall–Littlewood polynomial (Macdonald 1979, p104), namely $q^{r-1} P_{(r)}(x, q^{-1})$. It reduces to a power sum function in the case $q = 1$. The coefficients $\chi_\rho^\lambda(q)$ in (2) may be calculated from (3) and (4) by using the decomposition

$$s_\mu(x) s_\nu(x) = \sum_\sigma c_{\mu\nu}^\sigma s_\sigma(x) \tag{5}$$

where $c_{\mu\nu}^\sigma$ are the Littlewood–Richardson coefficients (Macdonald 1979, p68).

Wenzl (1988) and independently Kerov (1989), have investigated $H_n(q)$ when q is a primitive pth root of unity with $p \geqslant 3$. Their results are as follows.

(i) Although $H_n(q)$ is not semisimple there exist semisimple quotients $H_n^{(m,k)}(q)$ for each pair (m,k) with $m, k \geqslant 1$ and $p = m + k$.

(ii) The irreducible representations, $\pi_{[\lambda]}$, of $H_n^{(m,k)}(q)$ may be labelled by the elements, λ, of the set of all (m,k)-standard partitions of n defined by $P_n^{(m,k)} = \{\lambda \in P_n^m \mid \lambda_1 - \lambda_m \leqslant k\}$, where $P_n^m = \{\lambda \in P_n \mid \lambda'_1 \leqslant m\}$.

(iii) For each irreducible representation $\pi_{[\lambda]}$ of $H_n^{(m,k)}(q)$ with $\lambda \in P_n^{(m,k)}$ there exists an explicit construction of the corresponding irreducible module $V^{[\lambda]}$ with basis labelled by $T_\lambda^{(m,k)}$, the set of (m,k)-standard tableaux of shape λ.

As noted above the ring of symmetric functions plays a key role in the representation theory of the symmetric groups and $H_n(q)$ when q is not a root of unity. The generalization appropriate to the case where q is a root of unity is as follows (Goodman and Wenzl 1990). Let Λ^m denote $\mathbb{Z}[x_1, \ldots, x_m]^{S_m}$, and $I^{(m,k)}$ be the ideal generated by $\{s_\mu(x) \mid \mu_1 - \mu_m = k+1, \mu \in P^m = \cup_n P_n^m\}$. Define $\Lambda^{(m,k)} = \Lambda^m / I^{(m,k)}$ and

$$\hat{s}_\lambda(x) = s_\lambda(x) + I^{(m,k)}. \tag{6}$$

Just as $\{s_\lambda(x) \mid \lambda \in P^m\}$ is a $\mathbb{Z}$-basis for Λ^m, so $\{\hat{s}_\lambda(x) \mid \lambda \in P^{(m,k)} = \cup_n P_n^{(m,k)}\}$ is a $\mathbb{Z}$-basis for $\Lambda^{(m,k)}$. The coefficients in the decomposition

$$\hat{s}_\mu(x)\hat{s}_\nu(x) = \sum_{\sigma \in P^{(m,k)}} d^\sigma_{\mu\nu} \hat{s}_\sigma(x) \tag{7}$$

are generalized Littlewood–Richardson coefficients; their properties have been investigated by Goodman and Wenzl (1990). They show that reduction modulo $I^{(m,k)}$ may be accomplished through the action of an affine reflection group W. This group acts on $\mathbb{R}^m$ and is generated by S_m acting by permutation of coordinates and the translation $v \mapsto v + (m+k, 0 \ldots 0, -(m+k))$. The reduction then has the form

$$\hat{s}_\lambda(x) = \begin{cases} 0 & \text{if } w \cdot \lambda = \lambda \text{ for some } w \in W \text{ with } \epsilon(w) = -1 \\ \epsilon(w)\hat{s}_\sigma(x) & \text{if } w \cdot \lambda = \sigma \text{ for some } w \in W \text{ and } \sigma \in P_n^{(m,k)} \end{cases} \tag{8}$$

where $w \cdot \lambda = w(\lambda + \rho) - \rho$ with $\rho = (m-1, m-2, \ldots, 1, 0)$ and we have identified an element $\lambda \in P^m$ with the vector $(\lambda_1, \ldots, \lambda_m)$ in $\mathbb{R}^m$.

The coefficients $d^\sigma_{\mu\nu}$ are nothing other than the fusion coefficients that arise in conformal field theory (Verlinde 1988, Goodman and Nakanishi 1991). They may be evaluated in terms of Littlewood–Richardson coefficients using (5) and the modification procedure for reducing $\mathrm{mod}\ I^{(m,k)}$ (Goodman and Wenzl 1990, see also Kac 1989, Walton 1990, Cummins 1990) to give

$$d^\sigma_{\mu\nu} = \sum_{\lambda \in P^m} \epsilon^\sigma_\lambda c^\lambda_{\mu\nu} \tag{9}$$

where the coefficients $\epsilon^\sigma_\lambda \in \{0, 1, -1\}$, determined directly from (8), govern the fusion modification rule

$$\hat{s}_\lambda(x) = \epsilon^\sigma_\lambda \hat{s}_\sigma(x) \tag{10}$$

with $\lambda \in P^m$ and $\sigma \in P^{(m,k)}$. This rule is exemplified in table 1 for all $\lambda \in P_n^m$ with $1 \leqslant n \leqslant 5$ and $2 \leqslant m + k \leqslant 5$.

It is shown in Cummins and King (1992) that the Frobenius formula, analogous to (2), for the characters $\phi^\sigma_\mu(q)$ of the irreducible representations $\pi_{[\sigma]}$ of $H_n^{(m,k)}(q)$ with $\sigma \in P_n^{(m,k)}$ takes the form

$$\hat{p}_\rho(x; q) = \sum_{\sigma \in P_n^{(m,k)}} \phi^\sigma_\rho(q) \hat{s}_\sigma(x) \tag{11}$$

Table 1. Fusion modification rules: $\hat{s}_\lambda(x) = \epsilon^\sigma_\lambda \hat{s}_\sigma(x)$ for $\lambda \in P_n$ and $\sigma \in P^{(m,k)}_n$ with $1 \leqslant n \leqslant 5$, $2 \leqslant p = m + k \leqslant 5$, $m \geqslant 1$ and $k \geqslant 0$. The expression $\epsilon^\sigma_\lambda[\sigma]$ is displayed with each row and column labelled by $\{\lambda\}$ and (m, k), respectively.

$\{\lambda\}\backslash(m,k)$	$(1,1)$	$(2,0)$	$(1,2)$	$(2,1)$	$(3,0)$	$(1,3)$	$(2,2)$	$(3,1)$	$(4,0)$	$(1,4)$	$(2,3)$	$(3,2)$	$(4,1)$	$(5,0)$
$\{1\}$	$[1]$		$[1]$	$[1]$		$[1]$	$[1]$	$[1]$		$[1]$	$[1]$	$[1]$	$[1]$	
$\{2\}$	$[2]$	$-[1^2]$	$[2]$			$[2]$	$[2]$			$[2]$	$[2]$	$[2]$		
$\{1^2\}$		$[1^2]$		$[1^2]$			$[1^2]$	$[1^2]$			$[1^2]$	$[1^2]$	$[1^2]$	
$\{3\}$	$[3]$		$[3]$	$-[21]$	$[1^3]$	$[3]$				$[3]$	$[3]$			
$\{21\}$				$[21]$	$-[1^3]$		$[21]$				$[21]$	$[21]$		
$\{1^3\}$					$[1^3]$			$[1^3]$				$[1^3]$	$[1^3]$	
$\{4\}$	$[4]$	$[2^2]$	$[4]$	$-[2^2]$		$[4]$	$-[31]$	$[21^2]$	$-[1^4]$	$[4]$				
$\{31\}$		$-[2^2]$					$[31]$	$-[21^2]$	$[1^4]$		$[31]$			
$\{2^2\}$		$[2^2]$		$[2^2]$			$[2^2]$				$[2^2]$	$[2^2]$		
$\{21^2\}$								$[21^2]$	$-[1^4]$			$[21^2]$		
$\{1^4\}$									$[1^4]$				$[1^4]$	
$\{5\}$	$[5]$		$[5]$			$[5]$	$-[32]$	$[2^21]$		$[5]$	$-[41]$	$[31^2]$	$-[21^3]$	$[1^5]$
$\{41\}$				$-[32]$							$[41]$	$-[31^2]$	$[21^3]$	$-[1^5]$
$\{32\}$				$[32]$			$[32]$	$-[2^21]$			$[32]$			
$\{31^2\}$												$[31^2]$	$-[21^3]$	$[1^5]$
$\{2^21\}$								$[2^21]$				$[2^21]$		
$\{21^3\}$													$[21^3]$	$-[1^5]$
$\{1^5\}$														$[1^5]$

where $\hat{p}_\rho(x;q) = p_\rho(x;q) + I^{(m,k)}$ with $p_\rho(x;q)$ defined by (3) and (4), but now $x = (x_1, x_2, \ldots, x_m)$. This is a generalization of a result due to Kerov (1989) corresponding to the particular specialization $x = (1, q, q^2, \ldots, q^{m-1})$.

The generalization offered by (11) allows the characters $\phi^\sigma_\rho(q)$ to be evaluated in terms of the characters $\chi^\lambda_\rho(q)$ using the modification rules for reducing mod $I^{(m,k)}$. These rules imply that

$$\phi^\sigma_\rho(q) = \sum_{\lambda \in P_n} \epsilon^\sigma_\lambda \chi^\lambda_\rho(q) \tag{12}$$

with $\epsilon^\sigma_\lambda \in \{0, 1, -1\}$ as in (9) above (we set $\epsilon^\sigma_\lambda = 0$ if $\lambda'_1 > m$).

For example the character table for $H_3(q)$ in the generic case, for which q is not a root of unity, takes the form (King and Wybourne 1990, 1992)

$$\begin{array}{cc} & \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{array}{c} \{3\} \\ \{21\} \\ \{1^3\} \end{array} & \begin{pmatrix} 1 & q & q^2 \\ 2 & -1+q & -q \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

where the rows giving $\chi^\lambda_\rho(q)$ are labelled by $\{\lambda\}$ and the columns are labelled by the connectivity classes (ρ).

Now turning to the degenerate case for which $q^2 + q + 1 = 0$ and taking $m = 1$ and $k = 2$, the only $(1,2)$-standard partition of 3 is (3). From (12), since $\epsilon^3_3 = 1$,

Table 2. Irreducible characters $\phi^{\mu}_{\rho}(q)$, of $H_n(q)$ with q a primitive pth root of unity for $1 \leqslant n \leqslant 5$ and $2 \leqslant p \leqslant 5$. Columns are labelled by the subscripts appropriate to minimal words $g_{i_1} g_{i_2}, \cdots$ in the class ρ. Rows are labelled by $[\mu]$, and if $\mu \in P_n^{(m,k)}$ for some (m,k) with $p = m + k$ then the expression for ϕ^{μ}_{ρ}, as a linear combination of χ^{λ}_{ρ} has been given in the form $\sum_{\lambda \in P_n} \epsilon^{\mu}_{\lambda}\{\lambda\}$.

$p=2\ q=-1$	$[\mu]$	0	1	12	123	13	124	1234
$\{1\}$	$[1]$	1						
$\{2\}$	$[2]$	1	-1					
$\{3\}$	$[3]$	1	-1	1				
$\{21\}$	$[21]$	2	-2	1				
$\{4\}$	$[4]$	1	-1	1	-1	1		
	$[31]$	2	-2	1	0	2		
$\{5\}$	$[5]$	1	-1	1	-1	1	-1	1
	$[32]$	5	-5	3	-1	5	-3	0
	$[41]$	4	-4	3	-2	4	-3	1
$p=3\ q=\omega$	$[\mu]$	0	1	12	123	13	124	1234
$\{1\}$	$[1]$	1						
$\{2\}$	$[2]$	1	ω					
$\{1^2\}$	$[1^2]$	1	-1					
$\{3\}$	$[3]$	1	ω	$-1-\omega$				
$\{21\}-\{3\}$	$[21]$	1	-1	1				
$\{4\}$	$[4]$	1	ω	$-1-\omega$	1	$-1-\omega$		
	$[31]$	3	$-1+2\omega$	$-1-2\omega$	$1+\omega$	$-1-3\omega$		
$\{2^2\}-\{4\}$	$[2^2]$	1	-1	1	-1	1		
	$[21^2]$	3	$-2+\omega$	$1-\omega$	ω	$1-2\omega$		
$\{5\}$	$[5]$	1	ω	$-1-\omega$	1	$-1-\omega$	1	ω
	$[41]$	4	$-1+3\omega$	$-2-3\omega$	$2+\omega$	$-2-4\omega$	$3+2\omega$	-1
$\{32\}-\{41\}$	$[32]$	1	-1	1	-1	1	-1	1
	$[31^2]$	6	$-3+3\omega$	-3ω	$1+2\omega$	-5ω	$2+4\omega$	$-1-\omega$
	$[2^21]$	4	$-3+\omega$	$2-\omega$	$-1+\omega$	$2-2\omega$	$-1+2\omega$	$-\omega$

$\epsilon^3_{2,1} = 0$ and $\epsilon^3_{1^3} = 0$, we have

$$\phi^3_{\rho}(q) = \chi^3_{\rho}(q).$$

Similarly taking $m = 2$ and $k = 1$ the only $(2,1)$-standard partition of 3 is $(2,1)$. Since $\epsilon^{2,1}_3 = -1$, $\epsilon^{2,1}_{2,1} = 1$ and $\epsilon^{2,1}_{1^3} = 0$, we have

$$\phi^{21}_{\rho}(q) = -\chi^3_{\rho}(q) + \chi^{21}_{\rho}(q).$$

Combining these results, and using $q^2 + q + 1 = 0$ where appropriate, gives

$$\begin{array}{c c} & \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{array}{c} [3] \\ [21] \end{array} & \begin{pmatrix} 1 & q & -1-q \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

Table 2. (continued)

$p=4\ q=i$	$[\mu]$	0	1	12	123	13	124	1234
$\{1\}$	$[1]$	1						
$\{2\}$	$[2]$	1	i					
$\{1^2\}$	$[1^2]$	1	-1					
$\{3\}$	$[3]$	1	i	-1				
$\{21\}$	$[21]$	2	$-1+i$	$-i$				
$\{1^3\}$	$[1^3]$	1	-1	1				
$\{4\}$	$[4]$	1	i	-1	$-i$	-1		
$\{31\}-\{4\}$	$[31]$	2	$-1+i$	$-i$	$1+i$	$-2i$		
$\{2^2\}$	$[2^2]$	2	$-1+i$	$-i$	0	0		
$\{21^2\}-\{31\}+\{4\}$	$[21^2]$	1	-1	1	-1	1		
$\{5\}$	$[5]$	1	i	-1	$-i$	-1	$-i$	1
	$[41]$	4	$-1+3i$	$-2-i$	$1-i$	$-2-2i$	$2-i$	i
$\{32\}-\{5\}$	$[32]$	4	$-2+2i$	$-2i$	$1+i$	$-2i$	$1+i$	-1
	$[31^2]$	6	$-3+3i$	$-2i$	$1+i$	$-4i$	$2+2i$	-1
$\{2^21\}-\{32\}+\{5\}$	$[2^21]$	1	-1	1	-1	1	-1	1
	$[21^3]$	4	$-3+i$	$2-i$	$-1+i$	$2-2i$	$-1+2i$	$-i$
$p=5\ q=\zeta$	$[\mu]$	0	1	12	123	13	124	1234
$\{1\}$	$[1]$	1						
$\{2\}$	$[2]$	1	ζ					
$\{1^2\}$	$[1^2]$	1	-1					
$\{3\}$	$[3]$	1	ζ	ζ^2				
$\{21\}$	$[21]$	2	$-1+\zeta$	$-\zeta$				
$\{1^3\}$	$[1^3]$	1	-1	1				
$\{4\}$	$[4]$	1	ζ	ζ^2	ζ^3	ζ^2		
$\{31\}$	$[31]$	3	$-1+2\zeta$	$-\zeta+\zeta^2$	$-\zeta^2$	$-2\zeta+\zeta^2$		
$\{2^2\}$	$[2^2]$	2	$-1+\zeta$	$-\zeta$	0	$1+\zeta^2$		
$\{21^2\}$	$[21^2]$	3	$-2+\zeta$	$1-\zeta$	ζ	$1-2\zeta$		
$\{1^4\}$	$[1^4]$	1	-1	1	-1	1		
$\{5\}$	$[5]$	1	ζ	ζ^2	ζ^3	ζ^2	ζ^3	ζ^4
$\{41\}-\{5\}$	$[41]$	3	$-1+2\zeta$	$-\zeta+\zeta^2$	$-\zeta^2$	$-2\zeta+\zeta^2$	$-2\zeta^2$	$1+\zeta+\zeta^2$
$\{32\}$	$[32]$	5	$-2+3\zeta$	$-2\zeta+\zeta^2$	$-\zeta^2$	$1-2\zeta+2\zeta^2$	$\zeta-\zeta^2+\zeta^3$	0
$\{31^2\}-\{41\}+\{5\}$	$[31^2]$	3	$-2+\zeta$	$1-\zeta$	ζ	$1-2\zeta$	2ζ	$-1-\zeta$
$\{2^21\}$	$[2^21]$	5	$-3+2\zeta$	$1-2\zeta$	ζ	$2-2\zeta+\zeta^2$	$-1+\zeta-\zeta^2$	0
$\{21^3\}-\{31^2\}$ $+\{41\}-\{5\}$	$[21^3]$	1	-1	1	-1	1	-1	1

where the rows giving $\phi^{\sigma}_{\rho}(q)$ have been labelled by $[\sigma]$.

Alternatively it is possible to use (11) directly along with (10), in which case we have, for example

$$\begin{aligned}\hat{p}_{21}(q) &= (q\hat{s}_2 - \hat{s}_{1^2})s_1 = q\hat{s}_3 + (-1+q)\hat{s}_{21} - \hat{s}_{1^3} \\ &= (q\hat{s}_2)\hat{s}_1 = q\hat{s}_3 \bmod I^{(1,2)} \\ &= (-\hat{s}_{1^2})s_1 = -\hat{s}_{21} \bmod I^{(2,1)}.\end{aligned}$$

The coefficients of $\hat{s}_\sigma$ give the required characters. They coincide, as they must, with the entries in the second column of the two character tables given above.

It should be stressed that for $H_n(q)$ with q a primitive pth root of unity, the above analysis only yields the characters of those irreducible representations, $\pi_{[\sigma]}$, which are labelled by partitions σ of n with σ (m,k)-standard for some m and k such that $p = m + k$. In general these representations form a proper subset of the set of all the irreducible representations, $\pi_{[\mu]}$. These may be labelled by all partitions μ of n with μ p-regular (Dipper and James 1986), in the sense that no part μ_i of μ is repeated more than $p-1$ times. It so happens that for $n = 3$ and $p = 3$ the p-regular set coincides with the union of the $(1,2)$ and $(2,1)$-standard sets. This is no longer the case for $n = 4$ and $p = 3$, for example.

In the cases we have discussed for which σ is (m,k)-standard for some m and k the representation $\pi_{\{\sigma\}}$ is, in general, indecomposable but reducible. The module $V^{\{\sigma\}}$ contains an irreducible quotient module $V^{[\sigma]}$ which affords a realization of the irreducible representation $\pi_{[\sigma]}$ of $H_n(q)$. The basis of $V^{[\sigma]}$ is provided by a subset of the set T_σ of standard Young tableaux of shape σ. This subset $T_\sigma^{(m,k)}$ of T_σ is provided by those standard tableaux whose sequence of shapes obtained through the successive removal of the entries $n, n-1, \ldots, 1$ define partitions which are all (m,k)-standard (Kerov 1989, Wenzl 1989, Goodman and Wenzl 1990).

For example, in the case $n = 3$ and $\sigma = (21)$ the two standard tableaux

$$t_1^{21} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \text{and} \quad t_2^{21} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

define a basis, $v_{t_1^{21}}$ and $v_{t_2^{21}}$, of $V^{\{21\}}$. Correspondingly, for $p = 3$, the representation matrices take the form (King and Wybourne 1992)

$$\pi_{\{21\}}(g_1) = \begin{pmatrix} -1 & 1+q \\ 0 & q \end{pmatrix} \quad \text{and} \quad \pi_{\{21\}}(g_2) = \begin{pmatrix} 0 & q \\ 1 & -1+q \end{pmatrix}$$

where use has been made of the identity $1 + q + q^2 = 0$ in passing from the generic case to the case for which $q^3 = 1$.

The partition $\sigma = (21)$ is $(2,1)$-standard and $t_1^{21} \in T_{21}^{(2,1)}$, however $t_2^{21} \notin T_{21}^{(2,1)}$. It follows that there should exist a 1-dimensional irreducible representation $\pi_{[21]}$. This is confirmed by noting that there exists a submodule spanned by $v_{t_1^{21}} + v_{t_2^{21}}$, as can be seen by inspection of the above matrices, each of whose row sums is q. The irreducible quotient module $V^{[21]}$ is then recovered by imposing the identity

$$v_{t_2^{21}} = -v_{t_1^{21}} \quad \text{or equivalently} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} = - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}.$$

It follows that

$$\pi_{[21]}(g_1) = \pi_{[21]}(g_2) = (\,-1\,)\,.$$

More generally, for each p-regular partition μ, the recovery of $V^{[\mu]}$ from $V^{\{\mu\}}$ also involves forming the quotient with a submodule. The difficulty is that in cases for which μ is not (m,k)-standard for any m and k it is not even known which subset of

Table 3. Decomposition matrices, b^{λ}_{μ} of $H_n(q)$ with q a primitive pth root of unity for $1 \leqslant n \leqslant 5$ and $2 \leqslant p \leqslant 5$. The coefficients, defined by $\chi^{\lambda}_{\rho}(q) = \sum_{\mu} b^{\lambda}_{\mu} \phi^{\mu}_{\rho}(q)$ for $\lambda \in P_n$ and $\mu \in P_n$ with μ p-regular, are displayed in the form of a matrix whose rows and columns are labelled by $\{\lambda\}$ and $[\mu]$, respectively, along with the dimensions of $\pi_{\{\lambda\}}$ and $\pi_{[\mu]}$.

		1
	$p = 2$	[2]
1	{2}	1
1	$\{1^2\}$	1

		1	2
	$p = 2$	[3]	[21]
1	{3}	1	
1	$\{1^3\}$	1	
2	{21}		1

		1	2
	$p = 2$	[4]	[31]
1	{4}	1	
3	{31}	1	1
2	$\{2^2\}$		1
3	$\{21^2\}$	1	1
1	$\{1^4\}$	1	

		1	5	4
	$p = 2$	[5]	[32]	[41]
1	{5}	1		
5	{32}		1	
6	$\{31^2\}$	1	1	
5	$\{2^21\}$		1	
1	$\{1^5\}$	1		
4	{41}			1
4	$\{21^3\}$			1

		1	1
	$p = 3$	[3]	[21]
1	{3}	1	
2	{21}	1	1
1	$\{1^3\}$		1

		1	1	3	3
	$p = 3$	[4]	$[2^2]$	[31]	$[21^2]$
1	{4}	1			
2	$\{2^2\}$	1	1		
1	$\{1^4\}$		1		
3	{31}			1	
3	$\{21^2\}$				1

		1	4	4	1	6
	$p = 3$	[5]	$[2^21]$	[41]	[32]	$[31^2]$
1	{5}	1				
5	$\{2^21\}$	1	1			
4	$\{21^3\}$		1			
4	{41}			1		
5	{32}			1	1	
1	$\{1^5\}$				1	
6	$\{31^2\}$					1

		1	2	1	2
	$p = 4$	[4]	[31]	$[21^2]$	$[2^2]$
1	{4}	1			
3	{31}	1	1		
3	$\{21^2\}$		1	1	
1	$\{1^4\}$			1	
2	$\{2^2\}$				1

		1	4	1	4	4	6
	$p = 4$	[5]	[32]	$[2^21]$	[41]	$[21^3]$	$[31^2]$
1	{5}	1					
5	{32}	1	1				
5	$\{2^21\}$		1	1			
1	$\{1^5\}$			1			
4	{41}				1		
4	$\{21^3\}$					1	
6	$\{31^2\}$						1

		1	3	3	1	5	5
	$p = 5$	[5]	[41]	$[31^2]$	$[21^3]$	[32]	$[2^21]$
1	{5}	1					
4	{41}	1	1				
6	$\{31^2\}$		1	1			
4	$\{21^3\}$			1	1		
1	$\{1^5\}$				1		
5	{32}					1	
5	$\{2^21\}$						1

T_μ may be used to label an appropriate basis of the irreducible representation $V^{[\mu]}$. Nonetheless for small n and p it is possible to determine all possible submodules and corresponding quotients of the modules $V^{\{\lambda\}}$ for all partitions λ of n. The process may be carried out by hand using the matrices $\pi_{\{\lambda\}}$ tabulated previously (King and Wybourne 1992).

The outcome is not only an explicit realization of $\pi_{[\mu]}$ for all p-regular partitions μ but also a complete determination of the decomposition matrices whose elements b^μ_λ give the multiplicity of occurrence of $\pi_{[\mu]}$ as an irreducible constituent of $\pi_{\{\lambda\}}$. In addition the results lead to a complete determination of the irreducible characters $\phi^\mu_\rho(q)$ for all p-regular partitions μ. These irreducible characters are displayed in table 2 for all $\mu \in P_n$ with $1 \leqslant n \leqslant 5$ and $2 \leqslant p \leqslant 5$. The corresponding decomposition matrices are given in tables 3 for $n \leqslant p$. They conform with the block structure theorem established by Dipper and James (1987), and for $p = 2$ and 3 are entirely consistent with the much more extensive tabulation of James (1990).

References

Carter R W 1986 Representation theory of the 0-Hecke algebra *J. Algebra* **104** 89–103

Cummins C J 1990 su(n) and sp($2n$) WZW fusion rules *J. Phys. A: Math. Gen.* **24** 391–400

Cummins C J and King R C 1992 A Frobenius formula for A_{n-1} Hecke algebras at roots of unity *Preprint*

Dipper R and James G 1986 Representations of Hecke algebras of general linear groups *Proc. Lond. Math. Soc.* **52** 20–52

Dipper R and James G 1987 Blocks and idempotents of Hecke algebras of general linear groups *Proc. Lond. Math. Soc.* **54** 57–82

Goodman F M and Nakanishi T 1991 Fusion algebras in integrable systems in two dimensions *Phys. Lett.* **262B** 259–65

Goodman F M and Wenzl H 1990 Littlewood–Richardson coefficients for Hecke algebras at roots of unity *Adv. Math.* **82** 244–65

Gyoja A 1986 A q-analogue of Young symmetrizer *Osaka J. Math.* **23** 841–52

James G 1990 The decomposition matrices of $GL_n(q)$ for $n \leqslant 10$ *Proc. Lond. Math. Soc.* **60** 225–65

Jones V F R 1987 Hecke algebra representations of braid groups and link polynomials *Ann. Math.* **126** 335–88

Kac V 1989 Primary fields, fusion rules and homological algebra *Unpublished talk* Presented at the Can. Math. Soc. Seminar on Lie theory, differential equations and representation theory, CRM, Université de Montréal

Kerov S V 1989 Realizations of $*$-representations of Hecke algebras and Young's orthogonal form *J. Sov. Math.* **46** 2148–58

King R C and Wybourne B G 1990 Characters of Hecke algebras $H_n(q)$ of type A_{n-1} *J. Phys. A: Math. Gen.* **23** L1193–7

King R C and Wybourne B G 1992 Representations and traces of the Hecke algebras $H_n(q)$ of type A_{n-1} *J. Math. Phys.* **33** 4–14

Macdonald I 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)

Ram A 1991 A Frobenius formula for the characters of the Hecke algebras *Invent. Math.* **106** 461–88

Van der Jeugt J 1991 An algorithm for characters of Hecke algebras $H_n(q)$ of type A_{n-1} *J. Phys. A: Math. Gen.* **24** 3719–25

Verlinde E 1988 Fusion rules and modular transformations in 2D conformal field theory *Nucl. Phys.* **B 300** 360–76

Vershik A M and Kerov S V 1989 Characters and realizations of representations of an infinite-dimensional Hecke algebra, and knot invariants *Sov. Math. Dokl.* **38** 134–7

Walton M 1990 Algorithms for fusion rules: a proof *Phys. Lett.* **241B** 365

Wenzl H 1988 Hecke algebras of type A_n and subfactors *Invent. Math.* **128** 349–83, 11–14